

A Hopf algebra isomorphism between two realizations of the quantum affine algebra $U_q(\widehat{gl}(2))$

A.H. Bougourzi¹ and A. Sebbar²

¹ *Institute of Theoretical Physics*

SUNY at Stony Brook

Stony Brook, NY 11794

² *Department of Mathematics*

SUNY at Stony Brook

Stony Brook, NY 11794

Abstract

We consider the algebra isomorphism found by Frenkel and Ding between the RLL and the Drinfeld realizations of $U_q(\widehat{gl}(2))$. After we note that this is not a Hopf algebra isomorphism, we prove that there is a unique Hopf algebra structure for the Drinfeld realization so that this isomorphism becomes a Hopf algebra isomorphism. Though more complicated, this Hopf algebra structure is also closed, just as the one found previously by Drinfeld.

I Introduction

Besides their rich mathematical structures, quantum affine algebras play a crucial role in the exact calculation of some experimental physical quantities that have remained so far inaccessible through traditional methods [1, 2]. However, they are defined through various approaches and therefore it is important to find the relations among all of them. In particular, it is well known that quantum affine algebras admit two apparently different realizations in terms of currents. The first one which is based on the *RLL* formalism was derived by Reshetikhin and Semenov-Tian-Shansky [3], following the general Faddeev-Reshtikhin-Takhtajan matrix construction of quantum groups [4]. The second one was derived by Drinfeld as a quantum version of central extensions of loop affine algebras [5]. The natural question of finding the exact connection between the two arises then. Frenkel and Ding addressed this question and were able to construct through the Gauss decomposition an algebra isomorphism between them [6]. The crucial question of finding an isomorphism at the level of Hopf algebras remained unanswered. In fact, as we will see, for the Hopf algebras structures given in [6], the above isomorphism is not a Hopf algebra isomorphism. However, we show that this algebra isomorphism can be upgraded to a Hopf algebra isomorphism for a different but unique Hopf algebra structure of the Drinfeld realization. This provides a second example of a closed Hopf algebra structure for the latter realization.

II Brief review of the Frenkel-Ding isomorphism

Here we briefly recall the two realizations of $U_q(\widehat{gl(2)})$ and the algebra isomorphism between them as found by Frenkel and Ding [6].

II.1 The Reshetikhin-Semenov-Tian-Shansky realization of $U_q(\widehat{gl(2)})$

Since this realization is based on a matrix representation of quantum groups, and in particular on the $R(z)$ matrix which satisfies the Yang-Baxter equation with a spectral parameter z , let us denote this algebra by $U(R(z))$. We review this realization following Ref. [6] where a more compact but equivalent definition is given.

Definition 2.1 [3, 6] $U(R(z))$ is an associative algebra with a unit and is generated by the modes of the currents $\ell_{ij}^{\pm}(z) = \sum_{n=0}^{\infty} \ell_{ij}^{\pm}[\mp n]z^{\pm n}$, $i, j = 1, 2$; with defining relations:

$$\begin{aligned} \ell_{21}^{+}[0] &= \ell_{12}^{-}[0] = 0 \\ \ell_{ii}^{+}[0]\ell_{ii}^{-}[0] &= \ell_{ii}^{-}[0]\ell_{ii}^{+}[0] = 1, \quad i = 1, 2; \\ R(\frac{z}{w})L_1^{\pm}(z)L_2^{\pm}(w) &= L_2^{\pm}(w)L_1^{\pm}(z)R(\frac{z}{w}), \\ R(\frac{z_{+}}{w_{-}})L_1^{+}(z)L_2^{-}(w) &= L_2^{-}(w)L_1^{+}(z)R(\frac{z_{-}}{w_{+}}). \end{aligned} \tag{1}$$

Here the notations are as follows: $L^{\pm}(z) = (\ell_{ij}^{\pm}(z))_{i,j=1}^2$, $L_1^{\pm}(z) = L^{\pm}(z) \otimes 1$, $L_2^{\pm}(z) = 1 \otimes L^{\pm}(z)$, $z_{\pm} = zq^{\pm c/2}$ with q^c being the central element, and

$$R(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{(1-z)q}{1-zq^2} & \frac{z(1-q^2)}{1-zq^2} & 0 \\ 0 & \frac{1-q^2}{1-zq^2} & \frac{(1-z)q}{1-zq^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2}$$

Note also that in the above defining relations, $R(z)$ has a Taylor expansion in z (not in z^{-1}). The important feature of this algebra is that it is equipped with a Hopf algebra structure given by

$$\begin{aligned} \Delta(L^{\pm}(z)) &= L^{\pm}(zq^{\pm(1 \otimes c/2)}) \otimes L^{\pm}(zq^{\mp(c/2 \otimes 1)}), \\ S(L^{\pm}(z)) &= L^{\pm}(z)^{-1}, \\ \mathcal{E}(L^{\pm}(z)) &= I, \end{aligned} \tag{3}$$

where Δ , S and \mathcal{E} denote the comultiplication, the antipode and the counit respectively.

II.2 The Drinfeld realization of $U_q(\widehat{gl(2)})$

Here we briefly recall the defining relations of the Hopf algebra $U_q(\widehat{gl(2)})$ as a quantum version of the central extension of the loop algebra $\widehat{gl(2)}$.

Definition 2.2[5, 6]. This is an associative algebra with a unit and is generated by the modes of the currents $X^{\pm}(z) = \sum_{n \in \mathbb{Z}} X_n^{\pm} z^{-n}$ and $k_i^{\pm}(z) = \sum_{n=0}^{\infty} k_{i, \mp n}^{\pm} z^{\pm n}$, $i = 1, 2$; and the

central elements $q^{\pm\frac{1}{2}c}$, with defining relations:

$$\begin{aligned}
k_{i0}^+ k_{i0}^- &= k_{i0}^- k_{i0}^+ = 1, \\
k_i^\pm(z) k_i^\pm(w) &= k_i^\pm(w) k_i^\pm(z), \\
k_i^+(z) k_i^-(w) &= k_i^-(w) k_i^+(z), \\
\frac{z_\mp - w_\pm}{z_\mp q - w_\pm q^{-1}} k_1^\mp(z) k_2^\pm(w) &= k_2^\pm(w) k_1^\mp(z) \frac{z_\pm - w_\mp}{z_\pm q - w_\mp q^{-1}}, \\
k_1^\pm(z) X^+(w) k_1^\pm(z)^{-1} &= \frac{z_\pm q - w q^{-1}}{z_\pm - w} X^+(w), \\
k_2^\pm(z) X^+(w) k_2^\pm(z)^{-1} &= \frac{z_\pm q^{-1} - w q}{z_\pm - w} X^+(w), \\
k_1^\pm(z)^{-1} X^-(w) k_1^\pm(z) &= \frac{z_\mp q - w q^{-1}}{z_\mp - w} X^-(w), \\
k_2^\pm(z)^{-1} X^-(w) k_2^\pm(z) &= \frac{z_\mp q^{-1} - w q}{z_\mp - w} X^-(w), \\
(z q^{\pm 1} - w q^{\mp 1}) X^\pm(z) X^\pm(w) &= X^\pm(w) X^\pm(z) (z q^{\mp 1} - w q^{\pm 1}), \\
[X^+(z), X^-(w)] &= (q - q^{-1}) (\delta(z/w q^c) k_2^-(w_+) k_1^-(w_+)^{-1} - \delta(z q^c/w) k_2^+(z_+) k_1^+(z_+)^{-1}),
\end{aligned} \tag{4}$$

where the formal δ function is defined by $\delta(z) = \sum_{n \in \mathbb{Z}} z^n$. This algebra is equipped with a Hopf algebra structure. The comultiplication, the antipode and the counit are given by the following relations:

$$\begin{aligned}
\Delta(k_i^+(z)) &= k_i^+(z q^{-1 \otimes \frac{c}{2}}) \otimes k_i^+(z q^{\frac{c}{2} \otimes 1}), \\
\Delta(k_i^-(z)) &= k_i^-(z q^{1 \otimes \frac{c}{2}}) \otimes k_i^-(z q^{-\frac{c}{2} \otimes 1}), \\
\Delta(X^+(z)) &= X^+(z) \otimes 1 + k_2^-(z q^{c \otimes 1}) k_1^-(z q^{c \otimes 1})^{-1} \otimes X^+(z q^{\frac{c}{2} \otimes 1}), \\
\Delta(X^-(z)) &= 1 \otimes X^-(z) + X^-(z q^{1 \otimes c}) \otimes k_2^+(z q^{1 \otimes \frac{c}{2}}) k_1^+(z q^{1 \otimes \frac{c}{2}})^{-1}, \\
S(k_i^\pm(z)) &= k_i^\pm(z)^{-1}, \\
S(X^+(z)) &= -k_1^-(z) k_2^-(z)^{-1} X^+(z), \\
S(X^-(z)) &= -X^-(z) k_1^+(z) k_2^+(z)^{-1}, \\
\mathcal{E}(k_i^\pm(z)) &= 1, \quad \mathcal{E}(X^\pm(z)) = 0.
\end{aligned} \tag{5}$$

In the above comultiplication the central elements $q^{\pm\frac{c}{2}}$ act nontrivially on the modules V_1 or V_2 appearing in a tensor product $V_1 \otimes V_2$ according to whether they are written as $q^{\pm\frac{c}{2} \otimes 1}$ or $q^{\pm 1 \otimes \frac{c}{2}}$ respectively. Since both the above realizations of $U_q(\widehat{gl}(2))$ are equipped with Hopf algebra structures it is therefore important to investigate whether they are Hopf-algebra isomorphic to each other. As we will show later the algebra isomorphism between them found in Ref. [6] is not a Hopf algebra isomorphism.

II.3 An algebra isomorphism

Here we briefly recall the algebra isomorphism φ between the above two realizations [6]. It is based on the uniqueness of the Gauss decomposition of the $L^\pm(z)$ matrix operators. In Ref. [6] it is shown that the map $\varphi: U(R) \rightarrow U_q(\widehat{gl(2)})$ defined by

$$\begin{aligned}\varphi(e^+(z_-) - e^-(z_+)) &= X^+(z), \\ \varphi(f^+(z_+) - f^-(z_-)) &= X^-(z), \\ \varphi(k_i^\pm(z)) &= k_i^\pm(z),\end{aligned}\tag{6}$$

where the currents $e^\pm(z)$ and $f^\pm(z)$ being uniquely defined by the following Gauss decompositions of $L^\pm(z)$:

$$L^\pm(z) = \begin{pmatrix} 1 & 0 \\ e^\pm(z) & 1 \end{pmatrix} \begin{pmatrix} k_1^\pm & 0 \\ 0 & k_2^\pm \end{pmatrix} \begin{pmatrix} 1 & f^\pm(z) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} k_1^\pm(z) & k_1^\pm(z)f^\pm(z) \\ e^\pm(z)k_1^\pm(z) & e^\pm(z)k_1^\pm(z)f^\pm(z) + k_2^\pm(z) \end{pmatrix}, \tag{7}$$

is an algebra isomorphism. Here the currents $e^\pm(z)$, $f^\pm(z)$, and $k_i^\pm(z)$ have all Taylor expansions in z or z^{-1} as:

$$\begin{aligned}e^+(z) &= \sum_{n=1}^{\infty} e_n^+ z^{-n}, & e^-(z) &= \sum_{n=0}^{\infty} e_{-n}^- z^n, \\ f^+(z) &= \sum_{n=0}^{\infty} f_n^+ z^{-n}, & f^-(z) &= \sum_{n=1}^{\infty} f_{-n}^- z^n, \\ k_i^\pm(z) &= \sum_{n=0}^{\infty} k_{i,\mp n}^\pm z^{\pm n}.\end{aligned}\tag{8}$$

Now that the algebra isomorphism between the two realizations is settled, it is natural to try to complete this investigation at the level of a Hopf algebra isomorphism because, after all, the major importance of these two realizations is their being equipped with Hopf algebra structures.

III A Hopf algebra isomorphism

In this section, we find a new Hopf algebra structure for the Drinfeld realization such that φ becomes a Hopf algebra isomorphism.

Definition 3.1 A homomorphism $\psi: (A, m, \eta, \Delta, S, \mathcal{E}) \rightarrow (B, m', \eta', \Delta', S', \mathcal{E}')$ is a Hopf algebra isomorphism if ψ is an algebra isomorphism between (A, m, η) and (B, m', η')

satisfying

$$\begin{aligned}
(\psi \otimes \psi) \circ \Delta &= \Delta' \circ \psi, \\
\psi \circ S &= S' \circ \psi, \\
\mathcal{E} &= \mathcal{E}' \circ \psi.
\end{aligned} \tag{9}$$

Note that ψ determines uniquely the Hopf algebra structure for B if the Hopf algebra structure of A is given. Moreover, we have the following existence result:

Proposition 3.2 *Let A and B be two associative algebras with unit together with an algebra isomorphism $\psi : A \longrightarrow B$. If there exists a Hopf algebra structure on A , then B can be endowed with a Hopf algebra structure such that ψ becomes a Hopf algebra isomorphism.*

Note that in the proposition we do not assume that B has a priori a Hopf algebra structure.

If such a Hopf algebra structure for B exists then relations (9) must be satisfied. Then we set

$$\begin{aligned}
\Delta' &= (\psi \otimes \psi) \circ \Delta \circ \psi^{-1}, \\
S' &= \psi \circ S \circ \psi^{-1}, \\
\mathcal{E}' &= \mathcal{E} \circ \psi^{-1}.
\end{aligned} \tag{10}$$

By construction, Δ' is an algebra homomorphism $B \longrightarrow B \otimes B$, S' is an algebra antihomomorphism $B \longrightarrow B^{op}$ and \mathcal{E}' is an algebra homomorphism $B \longrightarrow \mathbb{C}$. And using the fact that ψ is an algebra isomorphism and that Δ , S and \mathcal{E} make A a Hopf algebra, one can check that Δ' , S' and \mathcal{E}' satisfy the axioms of a Hopf algebra for B .

As for the isomorphism φ of II.3, it is clear that it does not preserve the Hopf algebra structure. Indeed, let us consider $k_1^\pm(z)$ in both $U_q(\widehat{gl(2)})$ and $U(R)$: in $U_q(\widehat{gl(2)})$ we have $S'(k_1^\pm(z)) = k_1^\pm(z)^{-1}$, but in $U(R)$, using $S(L^\pm(z)) = L^\pm(z)^{-1}$, we find $S(k_1^\pm(z)) = k_1^\pm(z)^{-1} + f^\pm(z)k_2^\pm(z)^{-1}e^\pm(z)$. Since $\varphi(k_1^\pm(z)) = k_1^\pm(z)$, $\varphi \circ S \neq S' \circ \varphi$.

Now using the proposition, we establish a different Hopf algebra structure for the Drinfeld realization so that φ becomes a Hopf algebra isomorphism. We find closed formulas for Δ' , S' and \mathcal{E}' leading to a new Hopf algebra structure for $U_q(\widehat{gl(2)})$. This is possible because $k_1^\pm(z)$ and $k_2^\pm(z)$ are invertible using the fact that l_{ii}^\pm are invertible and using the Gauss decomposition.

Theorem 3.3 The isomorphism $\varphi : U(R(z)) \rightarrow U_q(\widehat{gl(2)})$ is a Hopf algebra isomorphism if the Drinfeld realization of $U_q(\widehat{gl(2)})$ is equipped with the following new Hopf algebra

structure:

$$\begin{aligned}
\Delta(k_1^\pm(z)) &= k_1^\pm(z_1^\pm) \otimes k_1^\pm(z_2^\mp) + k_1^\pm(z_1^\pm) f^\pm(z_1^\pm) \otimes e^\pm(z_2^\mp) k_1^\pm(z_2^\mp), \\
\Delta(k_2^\pm(z)) &= \sum_{n=0}^{\infty} (-1)^n k_2^\pm(z_1^\pm) f^\pm(z_1^\pm)^n \otimes e^\pm(z_2^\mp)^n k_2^\pm(z_2^\mp), \\
\Delta(e^\pm(z)) &= e^\pm(z_1^\pm) \otimes 1 + \sum_{n=0}^{\infty} (-1)^n k_2^\pm(z_1^\pm) f^\pm(z_1^\pm)^n k_1^\pm(z_1^\pm)^{-1} \otimes e^\pm(z_2^\mp)^{n+1}, \\
\Delta(f^\pm(z)) &= 1 \otimes f^\pm(z_2^\mp) + \sum_{n=0}^{\infty} (-1)^n f^\pm(z_1^\pm)^{n+1} \otimes k_1^\pm(z_2^\mp)^{-1} e^\pm(z_2^\mp)^n k_2^\pm(z_2^\mp), \\
S(k_1^\pm(z)) &= k_1^\pm(z)^{-1} + f^\pm(z) k_2^\pm(z)^{-1} e^\pm(z), \\
S(k_2^\pm(z)) &= k_2^\pm(z)^{-1} - f^\pm(z) k_2^\pm(z)^{-1} \{ \sum_{n=0}^{\infty} (-1)^n (k_1^\pm(z) f^\pm(z) k_2^\pm(z)^{-1} e^\pm(z))^n \} \\
&\quad \cdot k_1^\pm(z) k_2^\pm(z)^{-1} e^\pm(z), \\
S(e^\pm(z)) &= - \{ \sum_{n=0}^{\infty} (-1)^n (k_1^\pm(z) f^\pm(z) k_2^\pm(z)^{-1} e^\pm(z))^n \} k_1^\pm(z) k_2^\pm(z)^{-1} e^\pm(z), \\
S(f^\pm(z)) &= - f^\pm(z) k_2^\pm(z)^{-1} \{ \sum_{n=0}^{\infty} (-1)^n (f^\pm(z) k_2^\pm(z)^{-1} e^\pm(z) k_1^\pm(z))^n \} k_1^\pm(z), \\
\mathcal{E}(k_i^\pm(z)) &= 1, \quad \mathcal{E}(e^\pm(z)) = \mathcal{E}(f^\pm(z)) = 0.
\end{aligned} \tag{11}$$

For the purpose of simplifying the notation we have set $z_1^\pm = z q^{\pm \frac{\epsilon}{2} \otimes 1}$ and $z_2^\pm = z q^{\pm 1 \otimes \frac{\epsilon}{2}}$. The actions of Δ , S and \mathcal{E} on the currents $X^\pm(z)$ of $U_q(\widehat{gl(2)})$ are easily derived from the above expressions and relations (6).

We have found a method for extending an algebra isomorphism to a Hopf algebra isomorphism when one of the algebras is endowed with a Hopf structure. It would be interesting to address the existence of a Hopf algebra isomorphism between $U(R(z))$ and $U_q(\widehat{gl(2)})$ with the Drinfeld comultiplication. Moreover, the same question could be raised for the Drinfeld-Jimbo definition of $U_q(\widehat{gl(2)})$ by means of Chevalley generators [7, 8]. There also Drinfeld found an algebra isomorphism between the latter algebra and the loop realization of $U_q(\widehat{gl(2)})$. However, the coalgebra structure of the latter realization is not explicit.

IV Acknowledgments

The work of A.H.B. is supported by the NSF Grant # PHY9309888. A.H.B. would like to thank Y. Saint-Aubin and H. Aurag for stimulating discussions.

References

- [1] A.H. Bougourzi, M. Couture and M. Kacir, Phys. Rev. B54 (1996) 12669.
- [2] A. Abada, A.H. Bougourzi and B. Si-Lakhal, preprint ITP-SB-72-96.
- [3] N. Yu. Reshetikhin and M.A. Semenov-Tian-Shansky, Lett. Math. Phys. 19 (1990) 133.
- [4] L.D. Faddeev, N.Yu. Reshetikhin, and L.A. Takhtajan, Advanced Series in Mathematical Physics, Vol. 10, Singapore, World Scientific (1989) 299.
- [5] V.G. Drinfeld, Proc. ICM, Am. Math. Soc., Berkeley, CA, (1986) 789.
- [6] J. Ding and I.B. Frenkel, Commu. Math. Phys. 156 (1993) 277.
- [7] V.G. Drinfeld, Sov. Math. Doklady 32, (1985) 254.
- [8] M. Jimbo, Lett. Math. Phys., 10 (1985) 63.